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Abstract

We introduce the *Orthogonal Colouring Game*, in which two players alternately colour vertices (from a choice of $m \in \mathbb{N}$ colours) of a pair of isomorphic graphs while respecting the properness and the orthogonality of the colouring. Each player aims to maximise her *score*, which is the number of coloured vertices in the copy of the graph she owns.

The main result of this paper is that the second player has a strategy to force a draw in this game for any $m \in \mathbb{N}$ for graphs that admit a *strictly matched involution*.

An involution σ of a graph G is *strictly matched* if its fixed point set induces a clique and any non-fixed point $v \in V(G)$ is connected with its image $\sigma(v)$ by an edge. We give a structural characterisation of graphs admitting a strictly matched involution and bounds for the number of such graphs. Examples of such graphs are the graphs associated with Latin squares and sudoku squares.

Keywords: Orthogonal Colouring Game, orthogonal graph colouring,

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mutually orthogonal Latin squares, scoring game, games on graphs, strictly matched involution

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1. Introduction

In this paper, we consider the following *orthogonal colouring game*, denoted by $MOC_m(G)$. The board of the game consists of two initially uncoloured disjoint isomorphic copies G_A and G_B of a given graph G. Two players, Alice and Bob, with Alice beginning, alternately choose one of the two graphs G_A or G_B and colour an uncoloured vertex of this graph with a colour from the set $\{1, \ldots, m\}$ such that adjacent vertices receive distinct colours (*i.e.*, the partial colouring is *proper*) and the orthogonality of the graphs is not violated. *Orthogonality* means that if v, w are two different vertices in G whose copies v_A, w_A (in G_A) resp. v_B, w_B (in G_B) are coloured, then

$$(c(v_A), c(v_B)) \neq (c(w_A), c(w_B)), \tag{1}$$

where $c(x)$ denotes the colour of a vertex x. When no such move is possible any more, the game ends. Alice owns G_A and Bob owns G_B. The *score* of a player is the number of coloured vertices in the board the player owns. If, at the end, the scores of both players are equal, there is a *draw*, otherwise, the player with the higher score wins.

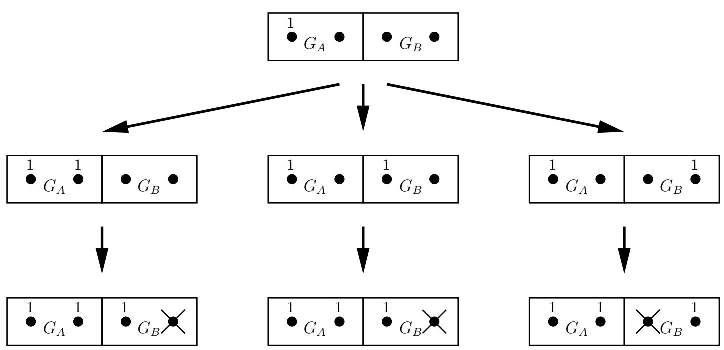


Figure 1: Alice's winning strategy for the game $MOC_1(2K_1)$

The main result of this paper states that for a special class of graphs, graphs admitting a *strictly matched involution*, the second player, Bob, can achieve at least a draw.

This class of graphs includes many special cases where the game is to create combinatorial objects such as orthogonal Latin rectangles, double diagonal Latin squares, Latin squares, and sudoku squares. However, there exist graphs in which optimal play from both players does not result in a draw. The smallest such example of a graph in which Alice wins is the graph $2K_1$ consisting of two isolated vertices with $m = 1$ colour, see Figure 1. An example of a case where Bob wins is $MOC_2(C_4)$. He wins as follows: when Alice plays on her first move on a C_4, Bob responds in the same C_4 on the non-adjacent vertex, colouring with the opposite colour if it is her C_4 and the same colour if it his C_4. Bob will then win by 2 points. Note that Bob's optimal strategy is not to play just in his graph.

Moreover, in an accompanying paper, Andres et al. [2] proved that it is PSPACE-complete to determine the outcome of the orthogonal colouring game for any $m \geq 3$ when an initial partial colouring is given.

Our paper is structured as follows. In Section 2, we motivate our research by the most prominent special case, the game played on the graph associated with Latin squares, and give references to related games and to some results on orthogonal graph colouring. In Section 3, we define graphs admitting a strictly matched involution and prove the main result of this paper. In Section 4, we study the graphs in which the game is a draw and prove for the most important special case, orthogonal Latin squares, that the game is a draw if $m = 1$. In Section 5, we provide a characterization of the graphs that admit a strictly matched involution which allows us to give an explicit construction for all such graphs and an upper bound for the number of these.

2. Motivation and Observations

The game $MOC_m(G)$ emanates from the overlap of two lines of research: combinatorial and scoring games (specifically, colouring games) and orthogonality of Latin squares or, more generally, of colourings of graphs.

Combinatorial games have been vastly studied (*e.g.*, the monographies of Albert et al. [1] or Berlekamp et al. [9], or, for a bibliography, the paper of Fraenkel [16]), particularly those where the board is a graph, *e.g.*, by Beaudou et al. [6], Beeler [8], or Faridi et al. [15]. In general, the loser is the first player who cannot move—called the *normal play* convention. Note that the drawing strategy for the second player in the orthogonal colouring game is a winning strategy for the second player in the normal play convention. In an accompanying paper, Andres et al. [2] proved that it is PSPACE-complete to determine the outcome of the game in the normal play convention when $m \in \mathbb{N}^*$ is the number of colours (even if m is a fixed constant), and an initial partial colouring is given.

The colouring game on graphs in the normal play convention, which was introduced as the ACHIEVEMENT game by Harary and Tuza [20] and called the PROPER k-COLOURING game by Beaulieu et al. [7], is closely related to the orthogonal colouring game on graphs. In the PROPER k-COLOURING game, two players take turns colouring the vertices of a graph, while maintaining that the colouring is proper. Beaulieau et al. [7] showed that this game is PSPACE-complete when $k \in \mathbb{N}^*$ is the number of colours (even if k is a fixed constant), and an initial partial colouring is given. For $k = 1$ colour, the PROPER k-COLOURING game is the well-studied game NODE KAYLES. For specific classes of graphs, it is known which player wins the game NODE KAYLES, *e.g.*, for paths and cycles a complete characterisation was given by Berlekamp et al. [9]. Harary and Tuza [20] characterised the winner in the PROPER k-COLOURING game played with $k = 2$ colours on paths and cycles, and played with any number k of colours on the Petersen graph. Astonishingly, as far as we know, the PROPER k-COLOURING game seems to not have been studied on other classes of graphs for $k \geq 2$ colours.

More recently, there has been the development of a theory of **scoring games** by Larsson et al. [22, 23, 24] where the winner is the one with the greater score. Another type of scoring games, sometimes also called *maker-breaker games*, are based on the interplay of minimising versus maximising a score. Here, game-theoretic graph parameters are motivated by trying to get good approximations to graph parameters that are hard to calculate, *e.g.*, chromatic number [17, 10, 14] and domination number [21]. Seo and Slater [26] give generic examples of how such parameters can be defined. Typically, two players choose vertices (or edges or other sub-objects) without violating a given property (*e.g.*, independence). The score is the number of vertices chosen where one player wants to maximise the number and the other to minimise it. Note that, from the players' point of view, it is difficult to determine the winner.

In particular, a game-theoretic version of the chromatic number, the *game chromatic number*, introduced by Bodlaender [10], and several of its possible variations have been extensively studied in the last three decades in more than 100 papers (see the partial surveys by Bartnicki et al. [5], Tuza and Zhu [27] or Dunn et al. [13] for some references). Upper bounds for the game chromatic number of many classes of graphs have been determined, *e.g.*, for forests by Faigle et al. [14], outerplanar graphs by Guan and Zhu [19], and planar graphs by Zhu [28]. However, the complexity of determining the game chromatic number of a graph in general is still an open problem.

Larsson et al. [25] extended the Maximiser/Minimiser approach, to use two graphs, G and H, usually, but not necessarily, isomorphic. One player, Left, is the maximiser on G but the minimiser on H and the other player, Right, has the reverse goals. The score is the number of pieces played on G minus the number played in H with Left winning if the score is positive and Right winning if the score is negative, and it is a draw if the score is 0.

Orthogonal colourings of graphs, *i.e.*, proper colourings of two isomorphic copies G_A and G_B of a graph respecting the orthogonality condition (1), have been studied as well (*e.g.*, by Archdeacon et al. [3], Ballif [4], or Caro and Yuster [12]). Caro and Yuster [12] studied the parameters $O_\chi(G)$ and $O_{\chi_k}(G)$ which are the minimum number of colours in any pair of orthogonal colourings of G, respectively, required such that there exist k mutually orthogonal colourings of G. Specifically, the graph versions of combinatorial objects associated with orthogonality were studied by Ballif [4] such as Latin squares and Latin rectangles.

Orthogonal Latin squares are natural combinatorial objects where there are two 'boards' and these form the basis of a specific orthogonal colouring game played on Latin squares.

Recall (see Brualdi [11]) that an $n \times n$ square, partially filled with entries taken from $\{1, 2, \dots, n\}$, has the *Latin property* if each row and column does not contain any repeated entries. A fully filled $n \times n$ square is a *Latin square* if each entry is an integer between 1 and n (inclusive) and each row and each column contains all n integers, which implies that the square has the Latin property. For a (partially filled) $n \times n$ square, X, let $c_X(i, j)$ be the (i, j) entry and $\emptyset$ if (i, j) is unfilled. Let A and B be (partially filled) $n \times n$ squares. Then A and B are *orthogonal* if in the list

$$((c_A(i, j), c_B(i, j)))_{\substack{1 \le i \le n \\ 1 \le j \le n}}$$

every ordered pair of integers occurs at most once. If A and B are Latin squares, this means that every pair of integers from $\{1, \dots, n\}^2$ occurs exactly once in the list.

Rules of the orthogonal Latin squares colouring game: There are two players, Alice and Bob, and two $n \times n$ squares labeled A and B. The squares are initially empty and Alice and Bob take turns filling one of the entries of either

A or B with an integer between 1 and m inclusive. After each move, both matrices must have the Latin property and the matrices must be orthogonal.

It is known that a Latin square of order n can be regarded as a proper colouring of the cartesian product of K_n with itself. Thus, the concept of orthogonal Latin squares translates easily to graph colourings and the orthogonal colouring game played on Latin squares is equivalent to $MOC_m(K_n \square K_n)$. See Figure 2 for an example of play.

3. Main Theorem

First, we fix some general notation.

For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$.

We use standard notation from graph theory. The *disjoint union* of two graphs H and H', denoted by $H \cup H'$, is the graph $(V \cup V', E \cup E')$ consisting of an isomophic copy (V, E) of H and an isomorphic copy (V', E') of H' with $V \cap V' = \emptyset$. The disjoint union $H \cup H$ of two identical graphs is also denoted by $2H$.

Recall that, for a graph $G = (V, E)$, an *automorphism* is a bijective mapping $\sigma : V \longrightarrow V$ with the property that

$$\forall v, w \in V : \quad (vw \in E \Longleftrightarrow \sigma(v)\sigma(w) \in E).$$

An *involution* of G is an automorphism σ of G with the property

$$\forall v \in V : \quad (\sigma \circ \sigma)(v) = v.$$

We define an involution of G to be *strictly matched* if

(SI 1) the set $F \subseteq V$ of fixed points of σ (*i.e.*, $F = \{v \in V \mid \sigma(v) = v\}$) induces a complete graph (*i.e.*, for every $v, w \in F$ with $v \neq w$ we have $vw \in E$) and

(SI 2) for every $v \in V \setminus F$, we have the (matching) edge $v\sigma(v) \in E$.

If, for a graph G, there exists a strictly matched involution, we say that G admits a strictly matched involution. The following is the main result of this paper.

Theorem 1. *Let G be a graph that admits a strictly matched involution and $m \in \mathbb{N}$. Then, the second player has a strategy guaranteeing a draw in the game $MOC_m(G)$.*

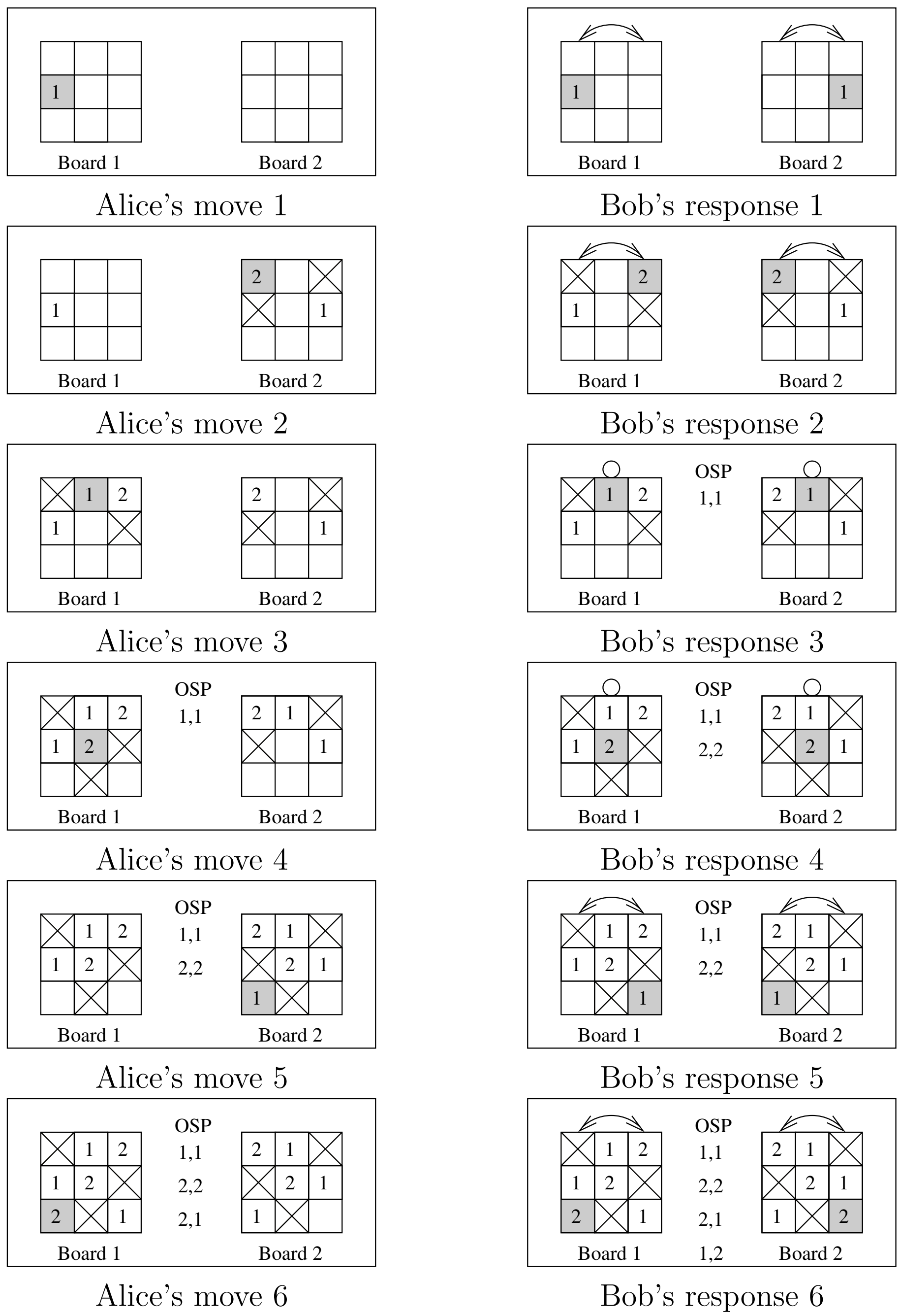


Figure 2: Bob's strategy from the proof of Theorem 1 guarantees a draw in the orthogonal Latin squares colouring game: an example played on 3×3 squares with 2 colours.

In Figure 2, we illustrate Bob's strategy given in the following proof of Theorem 1 on $K_3 \Box K_3$, where $K_3 \Box K_3$ is represented by a 3×3 board and the involution is given by the mirror symmetry around the middle column of each board. Bob's strategy on graphs in general just follows this idea using a strictly matched involution, as explained in the following.

Proof of Theorem 1. Let G_1 and G_2 be the two copies of $G = (V, E)$. For $k \in \{1, 2\}$, we denote by $c_k(v)$ the colour of the vertex $v \in V$ in G_k. In case the vertex v is uncoloured in G_k, we write $c_k(v) = \emptyset$. To simplify notation and differentiate between the colour of a vertex in a certain copy of G and an actual colour, we refer to the colours as symbols.

Let OSP be the set of *orthogonal symbol pairs*, *i.e.*, the set of those pairs (s_1, s_2) of symbols $s_1, s_2 \in [m]$, such that there exists a vertex $v \in V$ with

$$c_1(v) = s_1 \quad \text{and} \quad c_2(v) = s_2.$$

Let σ be a strictly matched involution, which exists by precondition.

For $m = 0$ or $|V| = 0$, the theorem is trivially true. Thus, assume $m, |V| \geq 1$. The strategy of the second player, Bob, is to copy (in a certain sense) Alice's moves in the other copy of the graph. Copying the symbols using the same positions would, in many cases, not be feasible because of orthogonality. Therefore, Bob couples the vertices of a graph with its image under σ of the other graph. Bob always plays the same symbol (=colour) as Alice just previously played.

For $c \in \{c_1, c_2\}$ we define $\overline{c}$ to be the other partial colouring from $\{c_1, c_2\}$ distinct from c.

Consider the case that Alice assigns $c(v) := s$ for some $c \in \{c_1, c_2\}$, some $v \in V$, and some symbol $s \in [m]$. Then, the copying strategy of Bob consists of assigning

$$\overline{c}(\sigma(v)) := s.$$

We will prove that Bob will force a draw with this strategy.

We observe, as a key of our analysis, the following invariants which hold for every $c \in \{c_1, c_2\}$, every $v \in V$, and every $s, s_1, s_2 \in [m]$ after each of Bob's moves:

1. Whenever $c(v) = s$, then $\overline{c}(\sigma(v)) = s$.
2. Whenever $c(v) = \emptyset$, then $\overline{c}(\sigma(v)) = \emptyset$.
3. Whenever $(s_1, s_2) \in OSP$, then $(s_2, s_1) \in OSP$.
4. Whenever $(s_1, s_2) \notin OSP$, then $(s_2, s_1) \notin OSP$.

We will prove by induction on the number of moves that after each move of Alice, Bob's move assigning $\overline{c}(\sigma(v)) = s$ according to his strategy is possible, *i.e.*,

a) the vertex $\sigma(v)$ is uncoloured in the colouring $\overline{c}$;

b) the move keeps the partial colourings being proper;

c) the move does not contradict the orthogonality of the colourings c_1 of G_1 and c_2 of G_2;

and that after each move of Bob, the invariants hold again.

At the beginning of the game, all invariants obviously hold.

Now consider a situation after a move of Alice, where she assigns $c(v) = s$. Therefore, before the move, vertex v was uncoloured, *i.e.*, we had $c(v) = \emptyset$. By invariant 2, we had $\overline{c}(\sigma(v)) = \emptyset$, thus, a) holds.

To prove b), assume to the contrary that the move of Bob would violate the properness of the partial colouring $\overline{c}$, *i.e.*, assume that there exists $w \in V$ with $w \neq \sigma(v)$ and $w\sigma(v) \in E$ such that

$$\overline{c}(w) = s = \overline{c}(\sigma(v)).$$

As Alice played on the vertex v with the partial colouring c, the assignment $\overline{c}(w) = s$ must have been made before her move. Then, by invariant 1, we have $c(\sigma(w)) = s$. But, since $w\sigma(v) \in E$ and σ is an involutive automorphism, we have

$$\sigma(w)v = \sigma(w)\sigma(\sigma(v)) \in E,$$

which contradicts $c(v) = s$, since from $w \neq \sigma(v)$ follows $v \neq \sigma(w)$ by the properties of σ.

To prove c), we remark the following. In the case Alice has created a new element $(s, x) \in OSP$, then by invariant 4, $(x, s) \notin OSP$ before Alice's move. Before proceeding with the proof, we will observe the following two key lemmas.

Lemma 2. *For all $x \in [m]$, at any time in the game, it is not possible for Alice to create a new element $(x, x) \in OSP$.*

Proof. Assume Alice created a new element $(x, x) \in OSP$. Then, by the definition of OSP, at some point in the game, a player has assigned $c_1(v_1) = x$ and, at some other point in the game, a player has assigned $c_2(v_1) = x$. At least one of these assignments was not performed in the last move w.r.t. the turn we consider. W.l.o.g. the assignment $c_1(v_1) = x$ was performed before the last move (the other case being symmetrical by interchanging the roles of G_1 and G_2). Then, by the invariants, the other player must have assigned in the same pair of moves

$$c_2(\sigma(v_1)) = x. \tag{2}$$

If $v_1 \in F$ (*i.e.*, v_1 is a fixed point of σ), then Bob created the new element $(x, x) \in OSP$.

Otherwise, *i.e.*, if $v_1 \in V \backslash F$, as already mentioned above, by the definition of OSP, at some point in the game a player has assigned

$$c_2(v_1) = x. \tag{3}$$

But (2) and (3) contradict the facts that, by the definition (SI 2) of a strictly matched involution, there is a matching edge $v_1\sigma(v_1)$ and, since c_2 is a proper partial colouring, v_1 and $\sigma(v_1)$ cannot be coloured the same colour. $\square$

Lemma 3. *In case Bob must create a new element $(x, x) \in OSP$ by his strategy, he is able to do so without violating orthogonality (i.e., $(x, x) \notin OSP$ before Alice's move).*

Proof. Assume $(x, x) \in OSP$ before Alice's move. Let v' be the vertex with $c_1(v') = x = c_2(v')$. By invariant 1 and by orthogonality, v' must be a fixed point of σ (*i.e.*, $v' \in F$). If Bob would have to create $(x, x) \in OSP$ for the second time, by his strategy, this would only be possible if Alice coloured a vertex $v'' \in F$, $v'' \neq v'$, with colour x. But this is impossible, since, by (SI 1), F induces a complete graph, thus, there is an edge $v'v'' \in E$, so that Alice could not have coloured v'' with the same colour as v'. Thus, the assumption is wrong. Therefore, Bob must create a new element (x, x) at most once. $\square$

We continue with the proof of c). By Lemma 2, after Alice's turn, we have

$$(s, x) \neq (x, s).$$

Therefore, also after Alice's move, $(x, s) \notin OSP$. Thus, the assignment

$$c_2(\sigma(v)) = s$$

of Bob is allowed (does not contradict orthogonality) and, since we have $c_1(\sigma(v)) = c_2(v) = x$, it will create a new element $(x, s) \in OSP$, satisfying invariant 3 and invariant 4.

In case Alice has created a new element $(x, s) \in OSP$, the arguments are the same (just interchange the roles of c_1 and c_2).

In case Alice does not create a new element in OSP on her move, Bob will have a feasible move by invariant 3 and invariant 4, and by reasons of symmetry, will not create a new element in OSP unless Alice played some symbol s on a vertex in F, in which case, Bob creates the new element $(s, s) \in OSP$, which maintains invariant 3 and invariant 4. The latter move of player 2 is feasible because of Lemma 3. This proves c).

Now consider a situation after the move of Bob and assume a), b), and c) to be true before his move. We have to prove that the 4 invariants hold again.

Within the proof of c), we have shown that after Bob's move, invariant 3 and invariant 4 hold again.

Invariants 1 and 2 follow from the definition of the assignment in Bob's move and the induction hypothesis.

This concludes the inductive step.

We have shown that Bob's strategy always allows a reaction to Alice's move. Therefore, the game will end before a move of the first player. In such a situation, Bob's copying strategy results in two partial colourings c_1 and c_2 with exactly the same number of coloured vertices. Thus, the game ends in a draw. $\square$

Corollary 4. *The second player has a strategy to guarantee a draw in orthogonal colouring games played on $n_1 \times n_2$ rectangles or $n \times n$ squares satisfying the Latin property (and possibly the double diagonal condition or the sudoku condition).*

To explain the notion used in Corollary 4: The *double diagonal condition* consists in demanding that the coloured entries of each of both diagonals in a square are pairwise different. The *sudoku condition* for an $n \times n$ square with $n = k^2$ and $k \in \mathbb{N}$ forces the coloured entries of each of the k^2 disjoint subsquares of size $k \times k$ to be pairwise different.

Proof of Corollary 4. For the graphs associated with such game boards, the assignment $(i, j) \longmapsto (i, n_2 + 1 - j)$, which describes a vertical mirror symmetry, is easily seen to be a strictly matched involution. $\square$

4. When the Game is a Draw

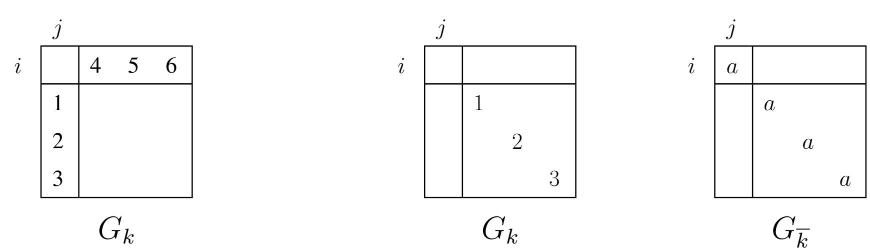
In this section, we look at graphs in general and also, at the special case of the graphs associated with Latin squares. We show that both players trivially have a strategy to draw if m is large enough. For the game $MOC_m(K_n \square K_n)$, we show that Alice has a strategy to draw if $m = 1$, thereby, showing that there exist graphs that admit a strictly matched involution where the optimal result for both players is a draw for some values of m.

First we note that, if m is large enough, both players have a strategy to force a draw. In the following lemma, for a graph G, the number $\Delta(G)$ is the maximum degree of a vertex in G and $\alpha(G)$ is the stability number (size of a maximum stable set) of G.

Lemma 5. *For any graph G and all $m \in \mathbb{N}$ with $m \geq \Delta(G) + \alpha(G)$, both players have a strategy to draw in the $MOC_m(G)$ game.*

Proof. We simply show that each of the players' copies, G_1 and G_2, of the graph G will be completely filled at the end of the game. To show this, we consider the worst possible case scenario for an uncoloured vertex v that needs to be coloured with some colour s in some copy G_k of G and show that it is possible to colour vertex v with s in G_k. For $k \in \{1, 2\}$, consider an uncoloured vertex v in some copy G_k of the graph G. Also, let $\overline{k} = 3 - k$. Thus, $G_{\overline{k}}$ is the other copy of the graph G that is not G_k.

In the case of the proper colouring property, the worst case is that every other vertex adjacent to v in G_k has been coloured with a distinct colour. Then, $\Delta(G)$ colours are unavailable to be played on v in G_k by the proper colouring property. See Figure 3 (a) for an example in the case $G = K_n \square K_n$.



(a) Latin property and (b) orthogonality restrictions

Figure 3: In the case the graph G is the graph $K_n \square K_n$: the worst case in the proof of Lemma 6 due to (a) the Latin property; (b) orthogonality, respectively.

By the orthogonality conditions, the worst case is that vertex v in $G_{\overline{k}}$ is coloured with some colour a and the forbidden pairs, (s_1, s_2) with $b = s_k$ and $a = s_{\overline{k}}$, exist for some $a \in [m]$ and $\alpha(G) - 1$ values of b, where $b \in [m]$.

Note that there cannot be more than $\alpha(G) - 1$ colours unavailable for v by the orthogonality conditions. This is because the colour a that appears in v of $G_{\overline{k}}$, may only appear at most $\alpha(G)$ times in $G_{\overline{k}}$ and hence, may only generate at most $\alpha(G) - 1$ forbidden pairs with G_k, since v is not coloured yet in G_k. See Figure 3 (b) for an example in the case $G = K_n \Box K_n$.

To be precise, the worst case is that each of the $\alpha(G) - 1$ unavailable colours by the orthogonality conditions for v in G_k, differ from each of the $\Delta(G)$ colours that are unavailable by the proper colouring property. Then, $\Delta(G) + \alpha(G) - 1$ colours are unavailable for v in G_k in the worst case. Therefore, since $m > \Delta(G) + \alpha(G) - 1$, the vertex v of G_k may always be coloured. $\square$

Corollary 6. *For all $m, n \in \mathbb{N}$ with $m \geq 3n-2$, both players have a strategy to draw in the $MOC_m(K_n \Box K_n)$ game.*

Proof. By Lemma 5, the result follows from the facts that $\alpha(K_n \Box K_n) = n$ and $\Delta(K_n \Box K_n) = 2n - 2$. $\square$

Lemma 7. *For all $n \in \mathbb{N}$, both players have a strategy to guarantee a draw in the $MOC_1(K_n \Box K_n)$ game.*

Proof. By Theorem 1, Bob has a strategy to force a draw.

Now, we show that Alice has a strategy to force a draw. Let G be $K_n \Box K_n$ and let G_k, $k \in \{1, 2\}$, be one copy of G and $G_{\overline{k}}$ be the other copy of G that is not G_k. We identify the graphs G_1 and G_2 with their underlying square boards. As we play with only $m = 1$ colour, it is easy to see that the only possible scores of both players are n and $n-1$, regardless of strategy. This is due to the fact that the orthogonality condition can only block at most one vertex in a row or column from being coloured, as otherwise, it would violate the Latin property. Therefore, as long as more than one possible vertex exists for a row or column in G_k, then one of the vertices can be coloured. Thus, the vertices of both G_k and $G_{\overline{k}}$ can be coloured until the point is reached where two rows and two columns have no coloured vertices in them and at least one of these vertices can be coloured, guaranteeing a score of at least $n - 1$.

Alice, who owns copy G_1 of G, colours a vertex in Bob's copy, G_2 of G, initially. Then, on every subsequent turn until there are $n - 2$ coloured vertices in G_2, Alice colours a vertex in $G_{\overline{k}}$ when Bob colours a vertex in G_k. Now, since Alice coloured a vertex in G_2 initially, eventually it is Bob's turn and there are $n-3$ coloured vertices in G_1 and $n-2$ coloured vertices in G_2. We show that Alice can force a draw from here. There are 3 cases based on the next move for Bob.

Case 1. *Bob colours a vertex in G_2 and there are no possible moves left in G_2.*

In this case, Bob achieved a score of $n-1$ and so Alice can at least draw if not win.

Case 2. *Bob colours a vertex in G_2 and there is still a possible move left in G_2.*

In this case, Alice colours the last colourable vertex in G_2 and Bob achieves a score of n. Bob is then forced to colour a vertex in G_1 and now it is Alice's turn. There are two rows and columns in G_1 with no coloured vertices in them, and no more vertices may be coloured in G_2. If none of the 4 colourable vertices remaining in G_1 are already coloured in G_2, then Alice will clearly achieve a score of n. Otherwise, at most 2 of the 4 colourable vertices remaining in Alice's board are already coloured in G_2 and, by the Latin property, they are not in the same row or column. Alice colours one of the 4 remaining colourable vertices in G_1 that is in the same row or column (but not the exact same position) as one of those at most 2 already coloured vertices in G_2. Now it is not possible to stop Alice getting a score of n since the last colourable vertex in G_1 is not coloured in G_2.

Case 3. *Bob colours a vertex in G_1.*

Both G_1 and G_2 have two remaining rows and columns with no coloured vertices in them. There are several cases of the possible situation. Let U_1 (U_2, respectively) be the set of the four possible remaining colourable vertices in G_1 (G_2, respectively). Let $[U_1]$ and $[U_2]$ be the preimage of U_1 and U_2 in G, respectively. Note that at most two of the copies of the vertices in U_1 may already be coloured in G_2 by the Latin property.

Subcase 3.1. *1 or 2 of the vertices in U_1 have the property that their copies are already coloured in G_2.*

If it is the case that two of the vertices in U_1 have this property, then these two vertices must be in different rows and columns since otherwise, the

Latin property would have been violated. Alice colours a vertex in U_1 that is in the same row or column (but not the exact same position) as one of these at most two already coloured vertices in G_2. It is clearly not possible to stop the last colourable vertex in G_1 from being coloured eventually which results in a score of n for Alice.

Subcase 3.2. *None of the copies of the vertices in U_1 have already been coloured in G_2.*

- If $[U_1] \cap [U_2] = \emptyset$, then clearly both players achieve a score of n.
- If $|[U_1] \cap [U_2]| \in \{1, 2\}$, then clearly Alice has a strategy to get a score of n by playing on a vertex in a position in $[U_1] \cap [U_2]$.
- The case $|[U_1] \cap [U_2]| = 3$ is not possible.
- If $[U_1] = [U_2]$, then Alice colours one of the vertices in U_1. If Bob colours a vertex in U_1, then Alice achieves a score of n and so at least draws the game if not wins. If Bob colours a vertex in U_2 in the same position or same column or row as the vertex Alice just coloured, then Alice can colour a vertex in U_1 and achieves a score of n and again, at least draws the game if not wins. Lastly, if Bob colours a vertex in U_2 but not in the same position nor the same column or row as the vertex Alice just coloured, then either Alice may still colour a vertex in U_1 if there are no forbidden pairs due to orthogonality yet, in which case Alice wins since Bob cannot colour a vertex in U_2 on the next turn by the orthogonality condition, or there is a forbidden pair due to orthogonality, in which case they draw with scores of $n - 1$ each since no vertices in U_1 nor U_2 can be coloured by the orthogonality condition.

□

5. Graphs that Admit a Strictly Matched Involution

We denote by $\mathcal{MI}$ the class of graphs that admit a strictly matched involution. See Figure 4 for a list of all graphs with at most 5 vertices that admit a strictly matched involution.

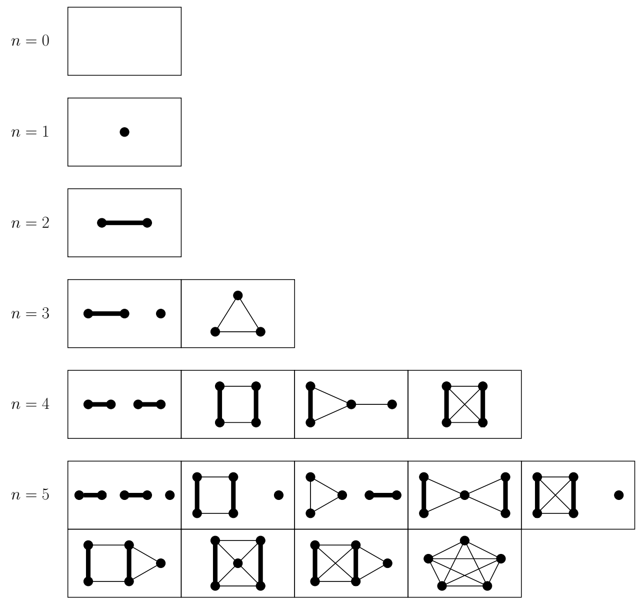


Figure 4: List of all graphs with ≤ 5 vertices that admit a strictly matched involution.

5.1. Characterising Graphs that Admit a Strictly Matched Involution

We first give an explicit characterization of all graphs $G \in \mathcal{MI}$. We then use this characterization to give an explicit construction for any graph $G \in \mathcal{MI}$. We note however, that in an accompanying paper, Andres et al. [2] proved that it is NP-complete to determine whether a graph admits a strictly matched involution.

Theorem 8. *A graph $G = (V, E)$ admits a strictly matched involution if and only if its vertex set V can be partitioned into a clique C and a set inducing a graph that has a perfect matching M such that:*

1. *for any two edges $vw, xy \in M$, the graph induced by v, w, x, y is isomorphic to*
   (a) *a $2K_2$ (2 disjoint copies of K_2) or*
   (b) *a C_4 (there are two possibilities for this) or*
   (c) *a K_4;*
2. *for any edge $vw \in M$ and any vertex $z \in C$, the graph induced by the vertices v, w, z is isomorphic to*
   (a) *a $K_1 \cup K_2$ or*
   (b) *a K_3.*

Proof. First, we prove the forward implication of the theorem, that is, if a graph $G = (V, E)$ admits a strictly matched involution, then the vertices V can be partitioned into a clique C and a matching M such that the properties (1.) and (2.) hold.

Thus, assume $G \in \mathcal{MI}$. Recall from the definition of a graph that admits a strictly matched involution, that (SI 1) and (SI 2) imply that the vertices V can be partitioned into a clique C and a matching M. Now, for any two edges $vw, xy \in M$, the graph induced by $v, w, x, y \in V$ is isomorphic to either:

- a $2K_2$ if no additional edges exist and note that this does not violate any conditions in the definition of a graph that admits a strictly matched involution.

- or a C_4 if vx and wy (vy and wx resp.) are edges in E or a K_4 if $vx, wy, vy, wx \in E$. Indeed, we prove that $vx \in E$ if and only if $wy \in E$ and $vy \in E$ if and only if $wx \in E$, thereby proving that a C_4 or a K_4 are the only two possibilities if additional edges exist. We prove the first case as the other is analogous. Since $G \in \mathcal{MI}$ by assumption, and therefore, by (SI 2) and since σ is an involution, $\sigma(v) = w$ and $\sigma(x) = y$, and since σ is an automorphism, $vx \in E \Leftrightarrow \sigma(v)\sigma(x) = wy \in E$.

For any edge $vw \in M$ and any vertex $z \in C$, the graph induced by $v, w, z \in V$ is isomorphic to either:

- a $K_1 \cup K_2$ if no additional edges exist and note that this does not violate any conditions in the definition of a graph that admits a strictly matched involution.

- or a K_3 if $vz, wz \in E$. Indeed, we prove that $vz \in E$ if and only if $wz \in E$, thereby proving that a K_3 is the only possibility if additional edges exist. The proof is analogous to the second case above and therefore, is omitted.

For the other implication, assume the vertices V can be partitioned into a clique C and a set inducing a graph that has a perfect matching M such that the properties (1.) and (2.) hold. We define a mapping σ as follows. For all vertices $z \in C$, let $\sigma(z) = z$ and for all edges $vw \in M$, let $\sigma(v) = w$ and $\sigma(w) = v$. We will prove that σ is a strictly matched involution.

Clearly, σ is involutive (*i.e.*, $\sigma(\sigma(v)) = v$ for every vertex $v \in V$) and (SI 1) and (SI 2) are satisfied. Now all that remains to show is that σ is a graph homomorphism. That is, it remains to be proven that

$$vw \in E \Longleftrightarrow \sigma(v)\sigma(w) \in E. \tag{4}$$

First, the forward direction of (4) is proven. Let $vw \in E$. If $vw \in M$, then by our mapping, $\sigma(v) = w$ and $\sigma(w) = v$ and we are done. So, assume $vw \notin M$. If $v, w \in C$, then $\sigma(v) = v$ and $\sigma(w) = w$ and we are done. So, w.l.o.g., assume that $v \notin C$ and let $vx \in M$. Then, $\sigma(v) = x$.

If $w \in C$, then $\sigma(w) = w$. Then, by property (2.), the graph induced by the vertices v, x, w is isomorphic to K_3 (since $vw \in E$ and $w \in C$) and hence, $xw = \sigma(v)\sigma(w) \in E$.

If $w \notin C$, then let $wz \in M$. Then, $\sigma(w) = z$. Since M is a matching, $z \notin \{v, x\}$. By property (1.), the graph induced by the vertices v, w, x, z is isomorphic to C_4 or K_4 (since $vw \in E$ and $vw \notin M$) and in either case, $\sigma(v)\sigma(w) = xz \in E$.

Using the forward direction and the fact that σ is involutive, we immediately get the backward direction of (4)

$$\sigma(v)\sigma(w) \in E \Longrightarrow vw = \sigma(\sigma(v))\sigma(\sigma(w)) \in E.$$

Thus, σ is strictly matched, *i.e.*, $G \in \mathcal{MI}$. □

Theorem 8 immediately implies the following structural result.

Corollary 9. *Any graph G on n vertices admitting a strictly matched involution has a partition of its vertex set into three (possibly empty) vertex subsets inducing a clique C of size $n - 2k$ and two isomorphic graphs H and H', each of size k, for some $k \in \mathbb{N}$ with $0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, respectively. Moreover,*

- *for any pair (v, v') of corresponding vertices $v \in V(H)$ and $v' \in V(H')$ and any vertex $w \in C$, either both vw and $v'w$ exist or none of them;*
- *for any pair (v, v') of corresponding vertices $v \in V(H)$ and $v' \in V(H')$, we have the existence of the matching edge $vv' \in E(G)$;*
- *for any two pairs (v, v') and (w, w') of corresponding vertices with $v, w \in V(H)$ and $v', w' \in V(H')$, either both vw' and $v'w$ exist or none of them.*

See Figure 5 for a sketch of the structure.

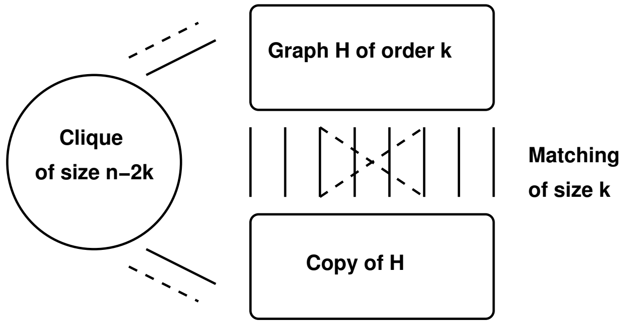


Figure 5: The structure of graphs admitting a strictly matched involution

According to Corollary 9, we can generate every graph on n vertices admitting a strictly matched involution if we fix some integer $k \leq \frac{n}{2}$ and take two copies of an arbitrary graph on k vertices which are matched by an isomorphism and add possible edges according to the rules given implicitly in Theorem 8 and explicitly in Corollary 9. Note that this construction may create isomorphic and even identical graphs. However, it gives us an upper bound for the number of such graphs, as specified in the next theorem (Theorem 11).

5.2. Counting Graphs that Admit a Strictly Matched Involution

In the following, let $g(n)$ be the number of isomorphism classes of graphs on n vertices. Let $A(n)$ be the number of isomorphism classes of graphs admitting a strictly matched involution on n vertices.

We use the following well-known fact.

Fact 10. *For any $n \in \mathbb{N}$,*

$$\frac{2^{\binom{n}{2}}}{n!} \leq g(n) \leq 2^{\binom{n}{2}}.$$

Theorem 11. *For any $n \in \mathbb{N}$,*

$$A(n) \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(k) 2^{\binom{k}{2}} 2^{(n-2k)k}.$$

Proof. To construct a graph $G \in \mathcal{MI}$ on n vertices, according to Corollary 9 (cf. Figure 5) we choose some nonnegative integer k with $k \leq \lfloor \frac{n}{2} \rfloor$ and a graph H on k vertices. By definition, for the latter, we have $g(k)$ choices. We add an isomorphic copy H' of H and a clique C on the remaining vertices.

For each pair of vertices of H we have two choices for the edges between these vertices and their copies in the copy H' of H: either there are only the two matching edges or all possible four edges exist. This gives us $2^{\binom{k}{2}}$ choices for the edges between H and H', since there are exactly $\binom{k}{2}$ such pairs.

For each pair (v, w) with $v \in V(H)$ and $w \in C$, we have two choices: either $vw \in E(G)$ or $vw \notin E(G)$. This gives us $2^{(n-2k)k}$ choices for the edges between H and the clique C. The edges between H' and C are then completely determined by these choices by Corollary 9.

Since all these choices are independent from each other, by summing over all k we get the claimed upper bound. □

Corollary 12. *For any $n \in \mathbb{N}$,*

$$A(n) \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{k(n-1-k)}.$$

Proof. By Theorem 11 and the right hand inequality of Fact 10, we obtain

$$A(n) \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(k) 2^{\binom{k}{2}} 2^{(n-2k)k} \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{\binom{k}{2}} 2^{\binom{k}{2}} 2^{(n-2k)k} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{(n-1-k)k}.$$

□

Corollary 13. *For any $n \in \mathbb{N}$,*

$$A(n) \leq 1 + \left\lfloor \frac{n}{2} \right\rfloor \cdot 2^{\left(\frac{n-1}{2}\right)^2}.$$

Proof. The exponent $f(k) = (n-1-k)k$ in Corollary 12 is maximised for $k = \frac{n-1}{2}$, therefore we conclude

$$A(n) \leq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} 2^{(n-1-k)k} = 1 + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 2^{(n-1-k)k} \leq 1 + \left\lfloor \frac{n}{2} \right\rfloor 2^{\left(\frac{n-1}{2}\right)^2}.$$

□

Corollary 14. *$A(n) = O\left(c(n)\sqrt{g(n)}\right)$ with $\log_2(c(n)) = o\left(\log_2 \sqrt[3]{g(n)}\right)$.*

Proof. Using Corollary 13 and the left hand inequality of Fact 10 we obtain

$$A(n) \leq 1 + \left\lfloor \frac{n}{2} \right\rfloor 2^{\left(\frac{n-1}{2}\right)^2} = 1 + \left\lfloor \frac{n}{2} \right\rfloor \frac{\sqrt{2^{\binom{n}{2}}}}{\sqrt{2^{\frac{n-1}{2}}}} \leq 1 + \frac{\left\lfloor \frac{n}{2} \right\rfloor \sqrt{n!}}{\sqrt{2^{\frac{n-1}{2}}}} \sqrt{g(n)}.$$

Then,

$$c(n) := \frac{\left\lfloor \frac{n}{2} \right\rfloor \sqrt{n!}}{\sqrt{2^{\frac{n-1}{2}}}}$$

is of the desired "*moderately exponential*" form, since $\log \sqrt{n!} = O(n \log n)$ (by Stirling's formula) but $\log \sqrt[3]{g(n)} = \Omega\left(n^2\right)$ (by Fact 10 and Stirling's formula). □

There is also a trivial lower bound, given in the next theorem.

Theorem 15. *For any $n \in \mathbb{N}$,*

$$A(n) \geq \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} g(k) 2^{(n-2k)k} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2}.$$

Proof. We construct a set of pairwise non-isomorphic graphs that admit a strictly matched involution on n vertices. For every $k = 0, 1, 2, 3, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$, take any graph H on k vertices and an isomorphic copy H' of H, and connect the vertices of H with their respective copy in H' but with no other vertex of H', and form a clique C with the remaining vertices and choose any type of connections between H and C and copy these connections between H' and C. This gives us in total

$$\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} g(k) 2^{(n-2k)k}$$

graphs. We claim that all of these, except for at most $\sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-2k}{2}$ pairs, are non-isomorphic.

Since all of the graphs H are non-isomorphic, two of the constructed graphs can only be isomorphic if a part of the clique can be considered as part of the matching. Since between the H and the H' of such a graph there are no edges apart from the matching edges, a part of the clique that is considered as part of the matching must consist of exactly two vertices v, w and these vertices must not be adjacent to any vertex in H and H' (as we forbid diagonal edges between H and H' in our construction). For fixed k, this gives us $\binom{|C|}{2} = \binom{n-2k}{2}$ pairs of vertices, resp. graphs, that have already occurred. Summing up (and observing that the bound is exact for $k = 0$), we get at most

$$\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2k}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2}$$

graphs for which an isomorphic graph was already constructed before. □

Corollary 16. *For any* $n \in \mathbb{N}$*,*

$$A(n) \geq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} 2^{\left(n-\frac{1}{2}\right)k - \frac{3}{2}k^2} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2}.$$

Proof. By Theorem 15 and the left hand inequality of Fact 10, we obtain

$$\begin{aligned} A(n) &\geq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} g(k) 2^{(n-2k)k} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} \\ &\geq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} 2^{\binom{k}{2}} 2^{(n-2k)k} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} 2^{\left(n-\frac{1}{2}\right)k - \frac{3}{2}k^2} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2}. \end{aligned}$$

□

Corollary 17. $A(n) = \Omega\left(d(n)\sqrt[3]{g(n)}\right)$ *with* $\log_2\left(\frac{1}{d(n)}\right) = o\left(\log_2 \sqrt[3]{g(n)}\right)$.

Proof. The exponent $f(k) = \left(n - \frac{1}{2}\right)k - \frac{3}{2}k^2$ in Corollary 16 is maximised for

$$k = \frac{n - \frac{1}{2}}{3},$$

therefore, for the integer value i in the interval $[(n-2)/3, (n+1)/3)$ we have

$$f(i) \geq f((n-2)/3).$$

Using this and the right hand inequality of Fact 10, we conclude from the lower bound in Corollary 16

$$\begin{aligned}
A(n) &\geq \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{k!} 2^{\left(n-\frac{1}{2}\right)k-\frac{3}{2}k^2} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} \\
&\geq \frac{1}{i!} 2^{\left(n-\frac{1}{2}\right)i-\frac{3}{2}i^2} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} \geq \frac{1}{n!} 2^{\left(n-\frac{1}{2}\right)i-\frac{3}{2}i^2} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} \\
&\geq \frac{1}{n!} 2^{\left(n-\frac{1}{2}\right)\frac{n-2}{3}-\frac{3}{2}\left(\frac{n-2}{3}\right)^2} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} = \frac{1}{n!} 2^{\frac{1}{3}\binom{n}{2}-\frac{1}{3}} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2} \\
&\geq \frac{\sqrt[3]{g(n)}}{n!\sqrt[3]{2}} - \left\lfloor \frac{n}{2} \right\rfloor \binom{n}{2}.
\end{aligned}$$

Then,

$$d(n) := \frac{1}{n!\sqrt[3]{2}}$$

is of the desired "*moderately exponential*" form since

$$\log(n!\sqrt[3]{2}) = O(n \log n)$$

(by Stirling's formula) but

$$\log \sqrt[3]{g(n)} = \Omega\left(n^2\right)$$

(by Fact 10 and Stirling's formula). □

Improvements on the lower bounds seem possible since in our construction in Theorem 15 we used only one of the $2^{\binom{k}{2}}$ possible ways to connect H with H'. We believe that the asymptotic behaviour of $A(n)$ is nearer to the upper bound than to the lower.

Conjecture 18. *Let* $m \in \mathbb{N}$. *Then*

$$A(n) = \Theta\left(c(n)\sqrt{g(n)}\right)$$

with $\log_2(c(n)) = o\left(\log_2 \sqrt[m]{g(n)}\right)$.

Conjecture 18 is justified by the observation that, in the construction of the graphs $G \in \mathcal{MI}$ in the proof of Theorem 11, relative to the total number of such graphs, not many pairs of graphs are isomorphic since most large graphs H have a trivial automorphism group (cf. Godsil and Royle [18, Chap. 2]).

6. Conclusion

We introduced a new scoring game on graphs called the orthogonal colouring game ($MOC_m(G)$). We have shown for a large class of graphs, *i.e.*, those that admit a strictly matched involution, that Bob has a strategy to force a draw. We have also shown that Alice has a strategy to force a draw in a subclass of this class of graphs, when the game is played with certain numbers m of colours. For further work, it would be interesting to find another class of graphs in which one of the players wins or the game is a draw. Specifically, for graphs admitting a strictly matched involution, it would be interesting to know when there is a winning strategy for the second player.

Problem 19. *Determine the outcome for the orthogonal colouring game for other classes of graphs.*

Problem 20. *For any $m \in \mathbb{N}$, characterise the class of graphs that admit a strictly matched involution for which the game with m colours is a draw (second player win, respectively).*

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